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RELATIVITY

THERE are two important questions to relativity: first, a physical question, "Do the laws of relativity hold true in the physical world?" and second, a mathematical question, "What are the laws of relativity?" The first of these will have to be decided by physical experiment, for at present it has not been decided nor is it likely to be for some time to come. The second was recently asked and answered by the Swiss scientist, Einstein,[1] without fear, though not without reproach.

Of the two questions the former is somewhat easier to discuss, and many a man is willing to discuss it with comparative indifference to the latter question. But it is the latter question which will concern us at present because a commentary on Einstein may be welcomed by some readers who like to know what things mean before deciding whether they are true, and because a study of relativity throws light on certain mathematical processes which are often obscured.

Mathematics may be defined as the science of arranging statements in consistent systems. For the sake of this consistency it is necessary that a mathematician should state as clearly as possible what his assumptions are. Then he must follow his assumptions to their logical conclusions, no matter where they may lead. But the mathematician is not a fatalist, and his procedure has the element of safety that if he does not like the conclusions, he is at liberty to change the assumptions.

His procedure is at the same time humble and arbitrary. It is humble because he calls an assumption an assumption, and arbitrary because he reserves the right to say to his reader, "Let us suppose that so-and-so is true"—although his reader may not wish to suppose that so-and-so is true at all, and, in fact, may be violently opposed to the assumption.

The most startling assumption of relativity is that the observed velocity of light is always constant, independent of the motion of the observer and the source of light. That is, light moves just as fast with respect to us, no matter whether we are moving toward its source or away from it, and no matter how fast we move. There is a single proviso that we should move, as indeed we are accustomed to move, with a velocity less than that of light. In order to bring out the meaning of this physical assumption, let us first propose a mathematical problem, How would a man alter the present ways of measuring time and space if he decided arbitrarily to record the velocity of light as constant, independent of the motion of the observer? A study of this problem throws light not only on the assumptions of physics, but also on the processes of mathematics, for the question involves the larger question, What does it mean to solve an equation?

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[1] *Annalen der Physik*, 1905, p. 891.

In early school life solving an equation means either finding an answer in the back of a book or satisfying an instructor that one has gone through a certain mechanical process whose logic, by the way, is apt to be questionable. If we solve an equation for x from this point of view we are supposed to be finding "the value of x." But there is no such thing as *the* value of x, since x may have any value. What we are really doing is finding, if possible, a value of x which will satisfy certain conditions. If the equation is $x - 5 = 0$, we go about its solution properly as follows: "Perhaps there is a value of x which, minus 5, is equal to 0, and if so, let us find it." Many students of mathematics do not understand that this is what it means to solve an equation, insomuch that the poet and story-teller, Edgar Allen Poe, is thoroughly annoyed and writes in his Purloined Letter "I never yet encountered the mere mathematician . . . who did not clandestinely hold it as a point of his faith that $x^2 + px$ was absolutely and unconditionally equal to q. Say to one of these gentlemen by way of experiment, if you please, that you believe occasions occur when $x^2 + px$ is not altogether equal to q, and having made him understand what you mean, get out of his reach as speedily as convenient, for beyond doubt he will endeavor to knock you down."

The "mere mathematician" described by Poe is evidently unqualified to express the laws of relativity, but the matter may safely be entrusted to three characters, *A, B,* and *C* that Stephen Leacock introduces to us in his *Literary Lapses* as follows:

"The student of arithmetic who has mastered the first four rules of his art, and successfully striven with money sums and fractions, finds himself confronted by an unbroken expanse of questions known as problems. These are short stories of adventure and industry with the end omitted, and, though betraying a strong family resemblance, are not without a certain element of romance.

"The characters in the plot of a problem are three people called *A, B,* and *C.* The form of the question is generally of this sort:

'*A*, *B*, and *C* do a certain piece of work. *A* can do as much work in one hour as *B* in two, or *C* in four. Find how long they work at it.'

"Or thus:

"'*A*, *B*, and *C* are employed to dig a ditch. *A* can dig as much in one hour as *B* can dig in two, and *B* can dig twice as fast as *C*. Find how long, etc., etc.'

"Or after this wise:

"'*A* lays a wager that he can walk faster than *B* or *C*. *A* can walk half as fast again as *B*, and *C* is only an indifferent walker. Find how far, and so forth.'

"The occupations of *A*, *B*, and *C* are many and varied. In the older arithmetics they contented themselves with doing 'a certain piece of work.' This statement of the case, however, was found too sly and mysterious, or possibly lacking in romantic charm. It became the fashion to define the job more clearly and to set them at walking-matches, ditch-digging, regattas, and piling cordwood. At times, they became commercial and entered into partnership, having with their old mystery a 'certain' capital. Above all they revel in motion. When they tire of walking-matches—*A* rides on horseback, or borrows a bicycle and competes with his weaker-minded associates on foot. Now they race on locomotives; now they row; or again they become historical and engage stage-coaches; or at times they are aquatic and swim. If their occupation is actual work they prefer to pump water into cisterns, two of which leak through holes in the bottom and one of which is watertight. *A*, of course, has the good one; he also takes the bicycle, and the best locomotive, and the right of swimming with the current. Whatever they do they put money on it, being all three sports. *A* always wins.

"Now to one who has followed the history of these men through countless pages of problems, watched them in their leisure hours dallying with cordwood, and seen their panting sides heave in the full frenzy of filling a cistern with a leak in it, they become something more than mere symbols. They appear as creatures of flesh and blood, living men with their own passions, ambitions, and aspirations like the rest of us. Let us view them in turn. *A* is a full-blooded blustering fellow, of energetic temperament, hot-headed and strong-willed. It is he who proposes everything, challenges *B* to work, makes the bets, and bends the others to his will. He is a man of great physical strength and phenomenal endurance. He has been known to walk forty-eight hours at a stretch, and to pump ninety-six. His life is arduous and full of peril. A mistake in the working of a sum may keep him digging a fortnight without sleep. A repeating decimal in the answer might kill him.

"B is a quiet, easy-going fellow, afraid of A and bullied by him, but very gentle and brotherly to little C, the weakling. He is quite in A's power, having lost all his money in bets.

"Poor C is an undersized, frail man, with a plaintive face. Constant walking, digging, and pumping have broken his health and ruined his nervous system. His joyless life has driven him to drink and smoke more than is good for him, and his hand often shakes as he digs ditches. He has not the strength to work as the others can, in fact, as Hamlin Smith has said, 'A can do more work in one hour than C in four.' "

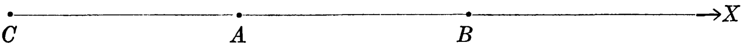
This arbitrary and headstrong A is just the man for our purpose. He is obsessed by the idea that the velocity of light is constant and he means to have it so regarded by his associates, B and C. In order to accustom B and C to his methods, A makes a few preliminary experiments in setting and regulating watches. For his first experiment A wishes to have B's watch always an hour behind C's, expressing standard time according to C's watch by values of a variable t and time according to B's watch by values of a variable τ. For his first experiment A establishes the relation between τ and t,

$$\tau = t - 1,$$

where τ is evidently a *function* of t; for a given value of t there is fixed by the equation a definite value of τ. As another experiment A wishes to have B's watch running only five-sixths as fast as C's (quite regardless of the inconvenience to B) and regulates B's watch so that B's time τ is again a function of C's time t,

$$\tau = 5/6\ t.$$

Now A is prepared to regulate his own watch and B's in another arbitrary way so that they will record the velocity of light as constant, independent of the observer's motion. A, B and C are disposed as follows: C is at rest; and A and B are in motion along the straight line CX with a constant velocity v. C's time, as before, is standard time and is represented by t; A's time and B's will be represented by τ. As A and B passed by C in their motion, each took care that his watch should agree with C's watch.



The problem in A's mind is to compare the time that it takes to flash a ray of light from A to B with the time that it takes to flash the ray back again from B to A. C argues that it evidently takes longer for the ray to pass from A to B than from B to A because B is moving away from the flash with velocity v, while A is coming up to meet it with the same velocity. But A says "I want that ray of light to measure the *same time* both from A to B and from B to A, and if C's way of measuring time and space won't show such a result, I am going to devise a system of my own that will.''

What A wants is to express his own time and B's by a function τ so that

$$\tau_1 - \tau_0 = \tau_2 - \tau_1, \tag{1}$$

where τ_0 is the time by A's watch when A sends the flash, τ_1 is the time by B's watch when B receives the flash and sends it back, and τ_2 is the time when A receives it again. The disturbing factor in the problem is the common velocity v of both A and B, therefore we may expect τ to be a function of v. And so it turns out;[2] τ is found to be a function of v and also of t, the time by C's watch since B and A left C, respectively. In terms of v and t the function τ is expressed by the equation

$$\tau = \frac{1}{\beta} \cdot t ; \; \beta > 1 \tag{2}$$

where $1/\beta$ is a function of v which is less than 1 and becomes smaller as v increases, approaching zero as v approaches the velocity of light. In other words, the faster A and B go the slower their watches must go.

Then from C's point of view no two of the three watches are together. If C's watch and B's watch were together, say, at twelve o'clock, when B left C, at a given time t, say half past two by C's watch, B's watch is behind C's in the ratio 1 to β and is regulated so as to get behind more and more in amount, but always in the same ratio. The like is true of A's watch, but inasmuch as B passed C before A did, B's watch, while regulated like A's watch, has been losing time longer, and so from C's point of view is behind A's watch. If B were to make his watch agree with A's (from C's point of view) he would have to set it ahead, but would not need to regulate it. If either A or B were to make his watch agree with C's (from C's point of view) he would have to set it ahead and regulate it to run faster.

But A is not satisfied with his first achievement recording the velocity of light as constant for both A and B when the light is flashed along the line CX. He wants to make further changes in his system of measurement so that the number which records the velocity of light shall be the same for C as well as for A and B and shall be the same whether the light is flashed along CX or perpendicular to CX or in any other direction. To solve this problem A must change his unit of length as well as his unit of time and alter his foot-rule as well as his watch.[3] He has to make his foot-rule shorter then C's when measuring in the direction of his motion CX, but when measuring at right angles to his motion, he may use the same foot-rule that C uses. This means that A has no longer a fixed unit for length independent of direction. He has one unit of length for one direction and another for another, but he records the velocity of light as constant and cheerfully puts up with all the attendant inconveniences.

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[2] Supplement, p. 35.

[3] Supplement, p. 40.

So far we have regarded the question of relativity as a mathematical question, and the laws of relativity as those equations which would have to hold if the velocity of light is to be recorded as constant, assuming the laws of the "old" physics. And so far there can be no question whatever of the validity of our results. If the "old" assumptions hold true for the physical world, it would be quite possible for an observer to set and regulate his watch and shorten his foot-rule so as to record the velocity of light as constant, that is, as a fixed number of "miles" per "second," where the mile and the second do not have their old meaning if one is in motion, but return to their old meaning if one returns to rest.

But now let us consider the assumption of relativity, which may be stated in this form: The whole material universe shares with A his obsession that the velocity of light is constant under any set of circumstances whatever, and this velocity will be recorded as constant by perfect mechanisms such as watches whether the observers are at rest or in motion. This means that watches in motion regulate themselves and foot-rules adjust themselves to attain the desired end. So when A comes to regulate B's watch and his own, he finds that it is not necessary; the watches have done it for him, and as for shortening his foot-rule, the foot-rule has anticipated him and has obligingly shortened itself. This means, of course, that the constitution of matter is affected by its motion. Is this true? Opinions differ; I suppose no one knows.

If the relativity assumption is true, some surprising things might happen. If it were possible for a man to throw his watch into space with the velocity of light at twelve o'clock and have it return to him when the college clock was striking three, his watch would remain at twelve during its motion, would begin running again as soon as it reached him and would remain three hours behind the college clock until he set it forward. If it were possible for a man to leave the earth and travel through free space at the velocity of light for fifty years and return again to the earth, he would find himself younger than his own children because all his physical processes would have stopped in the meantime by virtue of his velocity.

But although the results of the relativity theory are surprising, they are not absurd because until lately we have never been concerned with velocities approaching in any way the velocity of light.

It is quite possible that electrons may move in accordance with the theory. At any rate, modern physicists are convinced that electrons do not move according to the classical laws of physics. Physicists are still divided into two camps with respect to Einstein's theories. Mathematicians are not divided because mathematics does not ask whether the laws of relativity are true or not, but restricts itself to showing what the laws of relativity are and from what assumptions they are derived.

SUPPLEMENT

Let the fixed point C be taken as the origin of coordinates for the observer C, and let (standard) time according to C's system be recorded by values of a variable t, and the distance from C of a given point on the moving line AB be recorded by values of the variable x, measured in C's units of length. Let the time according to A's system be given by values of a variable τ, where τ is to be determined as a function of x and the velocity v so that

$$\tag{1} \tfrac{1}{2}(\tau_0 + \tau_2) = \tau_1,$$

provided τ_0 represents the time of starting from A toward B of a ray of light, τ_1 represents the time when this ray of light is reflected from B back toward A, and τ_2 represents the time when the ray of light reaches A again.

It must be kept in mind that A's problem of finding a function τ such that

$$\tfrac{1}{2}(\tau_0 + \tau_2) = \tau_1,$$

is a mathematical problem and does not involve any physical assumption whatever.

To determine this function $\tau(x, t)$ A finds it convenient to introduce a new variable x' such that

$$x' \equiv x - vt,$$

where x' as well as x is measured in C's system. Since x' records the distance from the moving point A to a moving point P on AB, x' is evidently independent of the velocity v and also of the time t that the body AB has been in motion, and, therefore,

$$\frac{\partial x'}{\partial t} = 0.$$

From C's point of view the time occupied by the ray in passing from A to B is $\frac{x'}{V-v}$ and the time occupied by the ray in passing from B to A is $\frac{x'}{V+v}$, where V is the velocity of light in C's system and x' is the coordinate corresponding to the point *B*. Then the equation (1) above may be written

$$(1')\quad \frac{1}{2}\left[\tau(0,\,t)+\tau\left(0,\,\left\{t+\frac{x'}{V-v}+\frac{x'}{V+v}\right\}\right)\right]=\tau\left(x',\,t+\frac{x'}{V-v}\right).$$

A's problem is to find, if possible, a function of x' and t that will satisfy (1′). Of course, it is possible that there is no such function, but, on the other hand, it is quite possible that such a function exists, and we will at least attack the problem as if it could be solved. Let us make the assumption that τ is a linear function of x' and t, whence, corresponding to a single value for x' and a single value for t there will be a single value for τ, and let us suppose further that the value of τ is zero when both x' and t are zero. This will make τ a linear homogeneous function of x' and t. Perhaps these assumptions will make the problem impossible, but perhaps not. At any rate, if we make these assumptions we may apply Euler's theorem for homogeneous functions, which may be stated for this special case

$$x'\frac{\partial\tau}{\partial x'}+t\frac{\partial\tau}{\partial t}=\tau(x',\,t).$$

By virtue of this theorem we may replace each one of the terms in (1′) by an expression involving the partial derivatives of τ with respect to x' and t. For example, we may replace the expression $\tau(0,\,t)$ by the expression $t\dfrac{\partial\tau}{\partial t}$, and similarly we may replace the expression

$$\tau\left(0,\,\left\{t+\frac{x'}{V-v}+\frac{x'}{V+v}\right\}\right)$$

by the expression

$$\left\{t+\frac{x'}{V-v}+\frac{x'}{V+v}\right\}\frac{\partial\tau}{\partial t}$$

and the expression

$$\tau\left(x',\,t+\frac{x'}{V-v}\right)$$

by the expression

$$x'\frac{\partial\tau}{\partial x'}+\left(t+\frac{x'}{V-v}\right)\frac{\partial\tau}{\partial t}.$$

Making these substitutions in (1′) we have the equation

$$(2)\qquad \frac{1}{2}\left(\frac{1}{V-v}+\frac{1}{V+v}\right)\frac{\partial\tau}{\partial t}=\frac{\partial\tau}{\partial x'}+\frac{1}{V-v}\frac{\partial\tau}{\partial t}$$

or

$$(3)\qquad \frac{\partial\tau}{\partial x'}+\frac{v}{V^2-v^2}\frac{\partial\tau}{\partial t}=0.$$

On the assumption that the function τ is linear and homogeneous, it is easy to find a function which will satisfy (3), namely,

$$\tau = a\left(t - \frac{v}{V^2 - v^2}x'\right), \tag{4}$$

where a is constant with respect to t and x'.

Of course, the proof of this mathematical pudding is in the eating, and the question is now, Have we actually found a function τ that will satisfy (1′)? By substitution in (1′) this is found to be the case and the problem is completely solved. It makes no difference what value is given to the constant a; any value for a will fulfil the required conditions, so that we are free to choose a to satisfy some other set of conditions.

A goes on to define the distance ξ along the line *CX* passed over by a ray of light in a given time τ by means of the equation

$$\xi = V\tau, \tag{5}$$

where V is the number which records the velocity of light in *C*'s system. By this means *A* keeps his own measure of the velocity of light in the direction *CX* equal to *C*'s measure of the velocity of light, and ξ, which has been defined as a function of τ, is readily fixed as a function of *C*'s length x'. If we substitute for τ its value from (4) we have the equation

$$\xi = aV\left(t - \frac{v}{V^2 - v^2}x'\right) \tag{6}$$

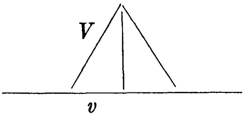
and since $t = \frac{x'}{V - v}$, equation (6) may be written

$$\xi = \frac{aV^2}{V^2 - v^2}x'. \tag{7}$$

A now defines the distance η perpendicular to *CX*, passed over by a ray of light in a given time τ by means of the equation $\eta = V \cdot \tau$ in order to keep the velocity of light equal to V when measured in the η-direction. Substituting for τ its value from (4) and noting that $x' = 0$,

$$\eta = aV \cdot t. \tag{8}$$

The quantity t may be expressed as a function of y as follows: From *C*'s point of view, a ray of light moving perpendicular to the line *CX* and reflected again so as to meet the observer *A* seems to have passed over the path of an isosceles triangle whose half base is proportional to v, whose leg is proportional to V, and whose altitude is y, so that the component of the velocity in the y-direction is $\sqrt{V^2 - v^2}$. The time t is given by the equation



$$\frac{y}{\sqrt{V^2 - v^2}} = t \tag{9}$$

and hence

$$\eta = \frac{aV}{\sqrt{V^2 - v^2}} y. \tag{10}$$

The form of (10) as well as (7) makes it convenient to denote the fraction $\frac{V}{\sqrt{V^2 - v^2}}$ by the symbol β, and to write the undetermined constant a as $\frac{1}{\beta}$ times a quantity $\phi(v)$ which is itself undetermined at present. Then (4), (7) and (10) may be written

$$\tau = \phi(v)\beta\left(t - \frac{v}{V^2}x\right), \tag{11}$$

$$\xi = \phi(v)\beta(x - vt), \tag{12}$$

$$\eta = \phi(v)y. \tag{13}$$

Let us now impose two conditions upon the undetermined function $\phi(v)$; first, that a transformation from C's system to A's, where A is regarded as having a velocity v with respect to C, followed by a transformation from A's system to a system that has a velocity $-v$ with respect to A, shall restore C's system unchanged. This will not interfere with the measurements of the velocity of light and will introduce symmetry into our systems—that is, C's system relative to A will behave as A's system relative to C.

If a transformation is made from C's system to A's, we have the equations (11), (12), and (13). Similarly if a transformation is made from A's system to a system moving away from A with the velocity $-v$, we have for the new system

$$t' = \phi(-v) \cdot \beta(-v) \cdot \left\{\tau + \frac{v}{V^2}\xi\right\}, \tag{14}$$

$$x'^{4} = \phi(-v)\beta(-v)(\xi + v\tau), \tag{15}$$

$$y' = \phi(-v)\eta. \tag{16}$$

Substituting in (14) the values of τ and ξ from (11) and (12) we

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[4] This use of x' must not be confused with the foregoing.

find that

$$t' = \phi(-v) \cdot \phi(v) \cdot t,$$

and, therefore, if C's system is to be restored unchanged by this double transformation, the product $\phi(-v) \cdot \phi(v)$ must be equal to 1. This imposes our first condition on the function $\phi(v)$.

Now let us impose on $\phi(v)$ the further condition that A's length η, measured in a direction perpendicular to his motion, shall bear the same relation to C's length y (parallel to η) as the coordinate η' of an observer A' who is moving away from C with A's velocity in the opposite direction. This again will not interfere with measurements of the velocity of light, but will introduce further symmetry into the given systems, since it will make the value of a coordinate perpendicular to the line of motion dependent only on the amount and not on the sense of the motion. If we impose this condition we have the equation

$$\eta = \eta', \tag{17}$$

and since

$$\eta = \phi(v) y, \tag{18}$$

$$\eta' = \phi(-v) y, \tag{19}$$

it follows that

$$\phi(v) = \phi(-v).$$

We have now imposed on the function $\phi(v)$ the two conditions $\phi(v) \cdot \phi(-v) = 1$ and $\phi(v) = \phi(-v)$, both of which are satisfied if we put $\phi(v)$ equal to 1. This value for $\phi(v)$ reduces our former equations (11), (12), and (13) to the following:

$$\tau = \beta\left(t - \frac{v}{V^2}x\right), \tag{20}$$

$$\xi = \beta(x - vt), \tag{21}$$

$$\eta = y, \tag{22}$$

where

$$\beta = \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}; \ \beta > 1,$$

and since, for the point A, $x = vt$ and for other points on CX, $x - vt = x'$ these equations may be written

$$\tau = \frac{1}{\beta}t \ ; \ \frac{1}{\beta} < 1, \tag{23}$$

$$\xi = \beta x' \ ; \ \beta > 1, \tag{24}$$

$$\eta = y. \tag{25}$$

This means that A's watch is running slower than C's and that his foot-rule, when measuring along CX is shorter than C's, but when measuring perpendicular to CX is the same as C's.

So far we have made no physical assumption. The physical assumption may be introduced at this point, namely, "The material universe behaves in accordance with the last group of equations."

Suppose now that a given mass m is moving in the direction CX with a given velocity. This velocity, measured by the observer C, is x'/t, and measured by the observer A, is ξ/τ. If we equate the momenta recorded by the two observers[5] we have the equation

$$m_C \cdot \frac{x'}{t} = m_A \cdot \frac{\xi}{\tau}; \quad \xi = \beta x', \quad \tau = \frac{1}{\beta} \cdot t.$$

From this equation we may find the mass from A's point of view in terms of the mass from C's point of view

$$m_A = \frac{1}{\beta^2} \cdot m_C.$$

Suppose that a given mass M is moving perpendicular to CX. Its velocity according to the observer C is y/t, and according to the observer A is η/t. If we equate the momenta recorded by the two observers, we have the relation

$$M_C \cdot \frac{y}{t} = M_A \frac{\eta}{\tau}; \quad \tau = \frac{1}{\beta} \cdot t, \quad \eta = y$$

from which we may find the mass from A's point of view in terms of the mass from C's point of view

$$M_A = \frac{1}{\beta} \cdot M_C.$$

The mass M is called by Einstein the *transverse* mass, the mass m the *longitudinal* mass of a given body. Both masses are functions of β and hence of A's velocity. Therefore, according to the Theory of Relativity, the measures of all three of the fundamental elements of physics, time, length, and mass vary with the velocity of the observer.

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[5] Cf. Bumstead, *Am. Jour. Science*, Vol. 4, No. 26, pp. 498–500.